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# The solution of the Hertz axisymmetric contact problem☆

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**Abstract**

The main terms of the asymptotic form of the solution of the contact problem of the compression without friction of an elastic body and a punch initially in point contact are constructed by the method of matched asymptotic expansions using an improved matching procedure. The condition of unilateral contact is formulated taking account of tangential displacements on the contact surface. An asymptotic solution of the problem for the boundary layer is constructed by the complex potential method. An asymptotic model is constructed, extending the Hertz theory to the case where the surfaces of the punch and elastic body in the vicinity of the contact area are approximated by paraboloids of revolution. The problem of determining the convergence of the contacting bodies from the magnitude of the compressive force is reduced to the problem of calculating the so-called coefficient of local compliance, which is an integral characteristic of the geometry of the elastic body and its fixing conditions.


## 1. Statement of the problem

Suppose a linearly elastic body occupies a three-dimensional region $\Omega$ with boundary $\partial\Omega = \Gamma_c \cup \Gamma_u \cup \Gamma_\sigma$. We will introduce a Cartesian system of coordinates with centre at the point $O$ on the boundary section $\Gamma_c$. To fix our ideas, we will assume that the $Ox_3$ axis is directed into the region $\Omega$, and here the plane $Ox_1x_2$ touches the surface $\Gamma_c$ at the point $O$.

We will assume that the surface of the body $\Omega$ is securely fixed on the boundary section $\Gamma_u$, and that on section $\Gamma_\sigma$ it is subjected to the action of surface loads with a density vector $\mathbf{q}(\mathbf{x})$. In such a case, the vector $\mathbf{u}(\mathbf{x})$ of displacements of points of the elastic body satisfies the relations

$$\mathcal{L}(\nabla_x)\mathbf{u}(\mathbf{x}) \equiv \mu\nabla_x \cdot \nabla_x\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\nabla_x\nabla_x \cdot \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{1.1}$$

$$\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_u \tag{1.2}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{x}) = \mathbf{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\sigma \tag{1.3}$$

where $\mathcal{L}(\nabla_x)$ is the Lamé operator, $\lambda$ and $\mu$ are Lamé parameters, and $\boldsymbol{\sigma}^{(n)}(\mathbf{x})$ is the stress vector on an area with unit vector of the outward normal $\mathbf{n}(\mathbf{x})$.

On the section $\Gamma_c$ the surface of the body $\Omega$ is in contact with an absolutely rigid body (the punch), the surface of which is specified by the equation $\Psi(\mathbf{x})=0$, where $\Psi(\mathbf{x})<0$ inside the punch and $\Psi(\mathbf{x})>0$ outside it. Bearing in mind

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the case of a punch in the form of a paraboloid with a convex surface $\Gamma_c$, where the contact area is localised in the vicinity of point $O$, on $\Gamma_c$ we will set more precise conditions of unilateral contact, formulated by Kravchuk,[1]

$$\Psi(\mathbf{x}) + \nabla_x\Psi(\mathbf{x})\mathbf{u}(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \Gamma_c \tag{1.4}$$

$$\Psi(\mathbf{x}) + \nabla_x\Psi(\mathbf{x})\mathbf{u}(\mathbf{x}) = 0 \Rightarrow \boldsymbol{\sigma}_T^{(n)}(\mathbf{x}) = 0, \quad \sigma_N^{(n)}(\mathbf{x}) \leq 0 \tag{1.5}$$

$$\Psi(\mathbf{x}) + \nabla_x\Psi(\mathbf{x})\mathbf{u}(\mathbf{x}) > 0 \Rightarrow \boldsymbol{\sigma}^{(n)}(\mathbf{x}) = 0 \tag{1.6}$$

where $\boldsymbol{\sigma}_N^{(n)} = \boldsymbol{\sigma}^{(n)}\mathbf{n}$ is the normal stress and $\sigma_T^{(n)} = \boldsymbol{\sigma}^{(n)} - \sigma_N^{(n)}\mathbf{n}$ is the shear stress vector.

The contact problem (1.1)–(1.6) is equivalent[1] to the problem of the minimum functional of potential energy on a set of admissible displacements satisfying conditions (1.2) and (1.4).

Below, we will consider the case

$$\Psi(\mathbf{x}) = x_3 + (2R_2)^{-1}(x_1^2 + x_2^2) - \delta_0 \tag{1.7}$$

where the constant $\delta_0 > 0$ determines the magnitude of the displacement of the punch.

We will also assume that the surface $\Gamma_c$ in the vicinity of point $O$ is mainly defined by the equation

$$x_3 = \Phi(x_1, x_2), \quad \Phi(x_1, x_2) = (2R_1)^{-1}(x_1^2 + x_2^2) \tag{1.8}$$

With the aim of using the perturbation method, we will introduce the small parameter $\varepsilon$, assuming that

$$\delta_0 = \varepsilon\delta_0^*, \quad \mathbf{q}(\mathbf{x}) = \varepsilon\mathbf{q}^*(\mathbf{x}) \tag{1.9}$$

Then, for small values of the parameter $\varepsilon$, using Hertz theory (see, for example, Ref. 2) the contact area $\omega_\varepsilon$ (unknown *a priori*) will mainly comprise a circular spot of diameter $O(\sqrt{\varepsilon})$.

The first refinement of the statement of the contact problem was proposed by Shtayerman. In Refs. 3,4 (see also Refs. 5,6), he extended Hertz theory to the case where the gap between the surfaces of elastic bodies initially in contact at a single point is defined mainly by the expression $A(x_1^2 + k^2x_2^2)^2$, where $A$ and $k$ are constants. Numerical methods were developed (see Refs. 7,8, etc.) for solving contact problems for an elastic body of finite dimensions. In cases where the body $\Omega$ can be replaced by a half-space, it is possible to use an approximate analytical solution[9] of the contact problem in a refined statement for which a numerical solution was obtained earlier.[10] In the case of a plane boundary section $\Gamma_c$, and without taking into account the tangential displacements in the contact area, an asymptotic solution was constructed.[11] Earlier, contact problems with spherical contact surfaces and a previously unknown interface of boundary conditions were investigated[12,13] using Aleksandrov's asymptotic method[14] (see also Ref. 15, Section 55).

In the present paper, using the method of matched asymptotic expansions[16,17] (see also the review Ref. 18) and an improved matching procedure,[19] the asymptotic form of the solution of the problem of unilateral contact without friction (1.1)–(1.6) with additional assumptions (1.7)–(1.9) and $\varepsilon \to 0+$ is constructed, and the first correction to the solution obtained by Hertz theory is written out in explicit form. The problem of determining the convergence of the contacting bodies from the magnitude of the compressive force is reduced to the problem of calculating the so-called coefficient of local compliance. The construction of the asymptotic form roughly follows the approach developed in Ref. 11.

## 2. The asymptotic form of Green's vector function with a pole on a curved boundary

We will denote by $\mathbf{G}(\mathbf{x})$ the solution of the homogeneous problem

$$\mathcal{L}(\nabla_x)\mathbf{G}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega; \quad \mathbf{G}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_u \tag{2.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_\sigma \cup \Gamma_c \backslash O \tag{2.2}$$

which possesses the following asymptotic expansion at the origin of coordinates

$$\mathbf{G}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + o(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to 0 \tag{2.3}$$

Here $\mathbf{T}(\mathbf{x})$ is the solution of the Boussinesq problem (see, for example, Ref. 20) concerning the action along the $Ox_3$ axis on an elastic half-space $x_3 \geq 0$ of a single concentrated force.

We recall that, in a cylindrical system of coordinates $r$, $\varphi$, $z$, in view of symmetry, $T_\varphi(r, \varphi, z) \equiv 0$, and the radial and vertical projections of the vector $\mathbf{T}(\mathbf{x})$ do not depend on the angle $\varphi$, and

$$4\pi\mu T_r(r, z) = \frac{rz}{s^3} - (1-2\nu)\frac{s-z}{rs}, \quad 4\pi\mu T_z(r, z) = 2(1-\nu)\frac{1}{s} + \frac{z^2}{s^3} \tag{2.4}$$

where $\nu$ is Poisson's ratio and $s = \sqrt{r^2 + z^2}$.

We will write the following term in asymptotic formula (2.3), due to the distortion of the surface $\Gamma_c$ in the vicinity of the point $O$. For this, we will investigate the behaviour as $\varepsilon \to 0$ of the solution $\mathbf{T}^\varepsilon(r, \varphi, z)$ of the Boussinesq secondary perturbed problem of the action at the origin of coordinates of a single concentrated force on an elastic semi-infinite body $z \geq \varepsilon\Phi_1(r)$, bounded by the surface

$$z = \varepsilon\Phi_1(r); \quad \Phi_1(r) = (2R_1)^{-1}r^2 \tag{2.5}$$

[Note that, in formula (2.5), $\varepsilon$ is a secondary parameter and bears no relation to (1.9).]

By virtue of the axial symmetry, we will have

$$\mathbf{T}^\varepsilon(r, \varphi, z) = T_r^\varepsilon(r, z)\mathbf{e}_r(\varphi) + T_z^\varepsilon(r, z)\mathbf{e}_3$$

where $\mathbf{e}_r(\varphi) = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2$ is the coordinate vector.

We will use the boundary shape perturbation method[16,21] (see also the papers on the mechanics of cracks Refs. 22,23). Applying obvious expansions (below, for simplicity, the dependence on the angular coordinate $\varphi$ is not indicated)

$$\mathbf{T}^\varepsilon(r, \varepsilon\Phi(r)) = \mathbf{T}^\varepsilon(r, 0) + \varepsilon\Phi(r)\mathbf{T}^\varepsilon_{,z}(r, 0) + O(\varepsilon^2)$$

$$\mathbf{n}^\varepsilon(r) = [1 + \varepsilon^2\Phi'(r)^2]^{-1/2}(\varepsilon\Phi'(r)\mathbf{e}_r - \mathbf{e}_3) = -\mathbf{e}_3 + \varepsilon\Phi'(r)\mathbf{e}_r + O(\varepsilon^2)$$

we obtain

$$-\boldsymbol{\sigma}^{(n)}(\mathbf{T}^\varepsilon; r, \varepsilon\Phi(r)) = \boldsymbol{\sigma}^{(z)}(\mathbf{T}^\varepsilon; r, 0) + \varepsilon[\Phi(r)\boldsymbol{\sigma}^{(z)}(\mathbf{T}^z_{,z}; r, 0) - \Phi'(r)\boldsymbol{\sigma}^{(r)}(\mathbf{T}^\varepsilon; r, 0)] + \ldots \tag{2.6}$$

Here

$$\boldsymbol{\sigma}^{(z)} = \sigma_{rz}\mathbf{e}_r + \sigma_{zz}\mathbf{e}_3, \quad \boldsymbol{\sigma}^{(r)} = \sigma_{rr}\mathbf{e}_r + \sigma_{rz}\mathbf{e}_3$$

The subscript $z$ after the comma denotes differentiation with respect to the $z$ coordinate, while differentiation with respect to $r$ is denoted by a prime.

Expansion (2.6) enables us to take the boundary condition

$$\boldsymbol{\sigma}^{(n)}(\mathbf{T}^\varepsilon; r, \varepsilon\Phi(r)) = 0, \quad r > 0 \tag{2.7}$$

on the unperturbed surface $z = 0$ of the elastic half-space. In fact, we will substitute into relation (2.7) the expansion (2.6) and the following

$$\mathbf{T}^\varepsilon(r, z) = \mathbf{T}(r, z) + \varepsilon\mathbf{t}(r, z) + \ldots \tag{2.8}$$

Collecting terms with the factor $\varepsilon$, to determine the second term of expansion (2.8) we derive the boundary condition

$$\boldsymbol{\sigma}^{(z)}(\mathbf{t}; r, 0) = -\Phi(r)\boldsymbol{\sigma}^{(z)}(\mathbf{T}_{,z}; r, 0) + \Phi'(r)\boldsymbol{\sigma}^{(r)}(\mathbf{T}; r, 0) \tag{2.9}$$

Direct calculations using formulae (2.4) with $z = 0$ and $r > 0$ give

$$\boldsymbol{\sigma}^{(z)}(\mathbf{T}_{,z}; r, 0) = 0, \quad \boldsymbol{\sigma}^{(r)}(\mathbf{T}; r, 0) = (1-2\nu)(2\pi)^{-1}r^{-2}\mathbf{e}_r$$

Substituting these expressions into the right-hand side of boundary condition (2.9), and recalling relation (2.5), we obtain

$$\sigma_{zz}(\mathbf{t}; r, 0) = 0, \quad \sigma_{rz}(\mathbf{t}; r, 0) = (1-2\nu)(2\pi R_1)^{-1}r^{-1} \tag{2.10}$$

We will construct the vector function $\mathbf{t}(r, \varphi, z)$ using the Papkovich–Neuber representation

$$t_r = \frac{\partial\varphi}{\partial r} + z\frac{\partial\psi}{\partial r}, \quad t_\varphi = 0, \quad t_z = \frac{\partial\varphi}{\partial z} - (3-4\nu)\psi + z\frac{\partial\psi}{\partial r} \tag{2.11}$$

On the basis of calculations,[24] taking expressions (2.10) into account, we find the harmonic potentials $\varphi(r, z)$ and $\psi(r, z)$ in the form

$$\varphi(r,z) = \frac{\vartheta(1-2\nu)}{R_1}\left(z\ln\frac{s+z}{2R_1} - s\right), \quad \psi(r,z) = \frac{\vartheta\beta}{2R_1}\ln\frac{s+z}{2R_1} \tag{2.12}$$

Here, we have used the notation

$$\vartheta = \frac{1-\nu}{2\pi\mu}, \quad \beta = \frac{1-2\nu}{1-\nu}$$

Hence, substituting expressions (2.12) into formulae (2.11), we will have

$$\begin{aligned} t_r(r,z) &= \frac{\vartheta\beta}{2R_1}\left(-2(1-\nu)\frac{r}{s+z} + \frac{rz}{s(s+z)}\right) \\ t_z(r,z) &= \frac{\vartheta\beta}{2R_1}\left(\frac{z}{s} - (1-2\nu)\ln\frac{s+z}{2R_1}\right) \end{aligned} \tag{2.13}$$

Let us return to the problem of refining the asymptotic form (2.3) of Green's vector function $\mathbf{G}(\mathbf{x})$. Taking into account assumption (1.8) concerning the approximation of the surface $\Gamma_c$ in the vicinity of the point $O$ by a circular paraboloid, on the basis of expansion (2.8) we establish the following:

$$\mathbf{G}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + \mathbf{t}(\mathbf{x}) + \tilde{\mathbf{A}} + O(|\mathbf{x}|\ln(|\mathbf{x}|/R_1)) \tag{2.14}$$

where $\mathbf{t}(\mathbf{x})$ is a vector function with cylindrical components (2.13), and $\tilde{\mathbf{A}}$ is a constant vector. Note that the quantity

$$A_0 = \vartheta^{-1}\tilde{\mathbf{A}}\mathbf{e}_3 \tag{2.15}$$

is an analogue of the corresponding coefficient of local compliance introduced earlier[25] for the case of a plane boundary $\Gamma_c$.

## 3. Outer asymptotic expansion

We will describe the structure of the solution $\mathbf{u}(\mathbf{x})$ of the initial problem (1.1)–(1.6) in the region $\Omega$ at a distance from the contact area $\omega_e$. The asymptotic expansion for the displacement field in the region indicated will be denoted by $\mathbf{v}(\mathbf{x})$.

Let $\mathbf{v}^0*(\mathbf{x})$ be the solution of the problem

$$\begin{aligned} &\mathcal{L}(\nabla_x)\mathbf{v}^0*(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega; \quad \mathbf{v}^0*(\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_u \\ &\boldsymbol{\sigma}^{(n)}(\mathbf{x}) = \mathbf{q}*(\mathbf{x}), \quad \mathbf{x}\in\Gamma_\sigma; \quad \boldsymbol{\sigma}^{(n)}(\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_c \end{aligned} \tag{3.1}$$

Then, by the method of matched asymptotic expansions, we will mainly have

$$\mathbf{v}(\mathbf{x}) = \varepsilon\mathbf{v}^0*(\mathbf{x}) + P\mathbf{G}(\mathbf{x}) \tag{3.2}$$

where $\mathbf{G}(\mathbf{x})$ is the solution of problem (2.1)–(2.3).

The quantity $P$, the dependence of which on the parameter $\varepsilon$ is not indicated for simplicity, is the equivalent of the contact pressures transferred to body $\Omega$ from the punch via the contact area $\omega_e$. It is clear that, because of the absence of friction [see Eq. (1.5)], the contact pressures act along a normal to the surface $\Gamma_c$.

The indeterminate coefficient $P$ is found by matching the outer asymptotic expansion (3.2) and the inner asymptotic expansion, the domain of validity of which covers the contact area. To establish the order of $P$ as $\varepsilon \to 0$, we will resort to the Hertz theory, which gives the main term of the asymptotic form of the solution of the contact problem.

Thus, free displacement of the point $O$ of the body $\Omega$ along the normal to the surface $\Gamma_c$ when there is no punch (due to the effect of surface loads on $\Gamma_\sigma$) would be a quantity $-\varepsilon\upsilon_3^{0*}(O)$. When the punch is indented to a depth $\delta_0$ (from a level $x_3 = 0$ along the $Ox_3$ axis directed into the body $\Omega$), mechanical work must be performed along a path of length $\varepsilon\delta_0^* - \varepsilon\upsilon_3^{0*}(O)$. Now, bearing in mind assumptions (1.7) and (1.8) concerning the axial symmetry of the punch surface and of the surface $\Gamma_c$ close to it, we obtain that the contact area $\omega_e$ is mainly circular with a radius

$$a \simeq \varepsilon^{1/2}((\delta_0^* - \upsilon_3^{0*}(O))R_0)^{1/2}; \quad R_0 = \frac{R_1R_2}{R_1 + R_2} \tag{3.3}$$

In this case, we obtain

$$P \simeq \varepsilon^{3/2}(\delta_0^* - \upsilon_3^{0*}(O))^{3/2} 4(3\pi)^{-1}\vartheta^{-1}R_0^{1/2} \tag{3.4}$$

Thus, relations (3.3) and (3.4) provide a basis for putting

$$a = \varepsilon^{1/2}a^*, \quad P = \varepsilon^{3/2}P^* \tag{3.5}$$

Relation (3.3) then dictates the introduction of extended coordinates with an extension factor $\varepsilon^{-1/2}$. In fact, by replacing the coordinates as follows:

$$\mathbf{x} = \varepsilon^{1/2}\boldsymbol{\xi}, \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \tag{3.6}$$

we obtain that the radius $a^*$ of the projection of the contact area onto the $O\xi_1\xi_2$ plane in the main term of the asymptotic form does not depend on the parameter $\varepsilon$.

The introduction of extended coordinates (3.6) enables us to describe the stress–strain state in the region of local perturbations (see Ref. 26, Section 133) under the base of the punch. In the intermediate region, the condition for matching of the outer and inner asymptotic expansions must be satisfied.[16] To derive this, asymptotic expansion of the vector function $\mathbf{v}(\mathbf{x})$ as $|\mathbf{x}| \to 0$ is necessary.

According to relation (2.14), we have

$$\mathbf{v}(\mathbf{x}) = P(\mathbf{T}(\mathbf{x}) + \mathbf{t}(\mathbf{x}) + \tilde{\mathbf{A}}) + \varepsilon\mathbf{v}^{0*}(O) + \varepsilon\sum_{k=1}^{6}\upsilon_{1,k}^{0*}\mathbf{V}^{1,k}(\mathbf{x}) + O\left(|\mathbf{x}|\ln\frac{|\mathbf{x}|}{R_1}\right) \tag{3.7}$$

where $\mathbf{V}^{1,k}(\mathbf{x})$ are linearly independent homogeneous vector polynomials of the first degree, satisfying in half-space $x_3 \geq 0$ the homogeneous Lamé system, and on its boundary the boundary conditions of no stresses (see, for example, Refs. 27,28).

Changing on both sides of this relation to extended coordinates (3.6), and taking into account the second relation of (3.5), after rearranging the terms we obtain

$$\mathbf{v}(\varepsilon^{1/2}\boldsymbol{\xi}) = \varepsilon P^*(\mathbf{T}(\boldsymbol{\xi}) + \varepsilon^{1/2}\mathbf{t}(\boldsymbol{\xi})) + \varepsilon\mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) + \dots \tag{3.8}$$

$$\mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) = \mathbf{v}^{0*}(O) + \varepsilon^{1/2}\left(P^*(\tilde{C}_\varepsilon\mathbf{e}_3 + \tilde{\mathbf{A}}) + \sum_{k=1}^{6}\upsilon_{1,k}^{0*}\mathbf{V}^{1,k}(\boldsymbol{\xi})\right) \tag{3.9}$$

Here

$$\tilde{C}_\varepsilon = \vartheta C_\varepsilon, \quad C_\varepsilon = -\frac{(1-2\nu)\beta}{2R_1}\ln\sqrt{\varepsilon} \tag{3.10}$$

Note that, in deriving expansion (3.8), we used explicit expressions (2.4) and (2.13) for the vectors $\mathbf{T}(\mathbf{x})$ and $\mathbf{t}(\mathbf{x})$. The estimate $O(\varepsilon^2|\boldsymbol{\xi}|\ln(\varepsilon|\boldsymbol{\xi}|/R_1))$ of the residual term in expansion (3.8) is determined by the error of formula (3.7).

## 4. Determination of the coordinates of the centre of the contact area

The inner asymptotic expansion for the displacement field in the immediate vicinity of the base of the punch will be constructed in extended coordinates (3.6) and denoted by $\mathbf{w}(\boldsymbol{\xi})$.

By the method of matched asymptotic expansions, the vector function $\mathbf{w}(\boldsymbol{\xi})$ is constructed in the semi-infinite region $\zeta \geq 1\sqrt{\varepsilon}\Phi_1(\rho)$ bounded by the surface

$$\zeta = \sqrt{\varepsilon}\Phi_1(\rho), \quad \Phi_1(\rho) = (2R_1)^{-1}\rho^2 \tag{4.1}$$

where $\zeta = \zeta_3$ and $\rho = \sqrt{\xi_1^2 + \xi_2^2}$ are extended cylindrical coordinates.

The condition for matching the inner asymptotic expansion $\mathbf{w}(\boldsymbol{\xi})$ and the outer asymptotic expansion $\mathbf{v}(\mathbf{x})$ according to expansion (3.8) dictates, for the vector $\mathbf{w}(\boldsymbol{\xi})$ in the matching zone (for$|\boldsymbol{\xi}| = \sqrt{\zeta^2 + \rho^2}\ \varepsilon^{-1/4}R_1$), the following asymptotic behaviour

$$\mathbf{w}(\boldsymbol{\xi}) = \varepsilon\mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) + \varepsilon P^*(\mathbf{T}(\boldsymbol{\xi}) + \varepsilon^{1/2}\mathbf{t}(\boldsymbol{\xi})) + O(\varepsilon^{7/4}|\ln\varepsilon|) \tag{4.2}$$

We will assume that

$$\mathbf{w}(\boldsymbol{\xi}) = \varepsilon\mathbf{V}^*(\varepsilon; \boldsymbol{\xi}) + \varepsilon\mathbf{W}^*(\boldsymbol{\xi}) \tag{4.3}$$

Since, the vector $\mathbf{V}^*(\varepsilon; \xi)$ by construction (see Eq. (3.9)) satisfies the Lamé system of equations, this will also be required of the vector function $\mathbf{W}^*(\boldsymbol{\xi})$.

Substituting the sum (4.3) into relation (4.2), we obtain

$$\mathbf{W}^*(\boldsymbol{\xi}) = P^*(\mathbf{T}(\boldsymbol{\xi}) + \varepsilon^{1/2}\mathbf{t}(\boldsymbol{\xi})) + O(\varepsilon^{3/4}|\ln\varepsilon|), \quad |\boldsymbol{\xi}| \sim \varepsilon^{-1/4}R_1 \tag{4.4}$$

The boundary condition of unilateral contact (1.4)–(1.6), written with respect to the vector function $\mathbf{w}(\boldsymbol{\xi})$, will be extended to the entire surface (4.1). Then, after the replacement of coordinates (3.6), condition (1.4) takes the form

$$\varepsilon\left(\frac{\rho^2}{2R_0} - \delta_0^*\right) + w_3\left(\xi_1, \xi_2, \frac{\sqrt{\varepsilon}}{2R_1}\rho^2\right) + \frac{\sqrt{\varepsilon}}{R_2}\sum_{i=1,2}\xi_i w_i\left(\xi_1, \xi_2, \frac{\sqrt{\varepsilon}}{2R_1}\rho^2\right) \geq 0 \tag{4.5}$$

Resorting again to the method of boundary shape perturbation, we take the boundary condition (4.5) on the plane $\xi_3 = 0$ which, in the limit, as $\varepsilon \to 0$, is approximated by paraboloid (4.1). Thus, leaving only the first correction, i.e. terms $O(\sqrt{\varepsilon})$ compared with unity, we obtain

$$\varepsilon\left(\frac{\rho^2}{2R_0} - \delta_0^*\right) + w_3(\xi_1, \xi_2, 0) + \frac{\sqrt{\varepsilon}}{2R_1}\rho^2\frac{\partial w_3}{\partial \xi_3}(\xi_1, \xi_2, 0) + \frac{\sqrt{\varepsilon}}{R_2}\sum_{i=1,2}\xi_i w_i(\xi_1, \xi_2, 0) \geq 0 \tag{4.6}$$

On the contact area $\omega^*$, relation (4.6) with the equality sign must be satisfied. Therefore, for all points $(\xi_1, \xi_2) \in \omega^*$, according to concept (4.3), we will have

$$\begin{aligned} &W_3^*(\xi_1, \xi_2, 0) + \frac{\sqrt{\varepsilon}}{2R_1}\rho^2\frac{\partial W_3^*}{\partial \xi_3}(\xi_1, \xi_2, 0) + \frac{\sqrt{\varepsilon}}{R_2}\sum_{i=1,2}\xi_i W_i^*(\xi_1, \xi_2, 0) = \\ &= \delta_0^* - \frac{\rho^2}{2R_0} - V_3^*(\varepsilon; \xi_1, \xi_2, 0) = \Delta_0^* - \frac{(\xi_1 - \xi_1^\circ)^2 + (\xi_2 - \xi_2^\circ)^2 + O(\varepsilon)}{2R_0} \end{aligned} \tag{4.7}$$

Here, we have used the equality

$$V_3^*(\varepsilon; \xi_1, \xi_2, 0) = \upsilon_3^{0*}(O) + \sqrt{\varepsilon}(\upsilon_{1,1}^{0*}\xi_2 - \upsilon_{1,2}^{0*}\xi_1) + \sqrt{\varepsilon}P^*(\tilde{C}_\varepsilon + \tilde{A}_0) \tag{4.8}$$

written taking into account definition (3.9), and the following notation is introduced

$$\xi_1^\circ = R_0\sqrt{\varepsilon}\upsilon_{1,2}^{0*}, \quad \xi_2^\circ = -R_0\sqrt{\varepsilon}\upsilon_{1,1}^{0*}, \quad \Delta_0^* = \delta_0^* - \upsilon_3^{0*}(O) - \sqrt{\varepsilon}P^*(\tilde{C}_\varepsilon + \tilde{A}_0) \tag{4.9}$$

The coefficients $\upsilon_{1,1}^{0*}$ and $\upsilon_{1,2}^{0*}$ have the meaning of small angles of rotation (about the coordinate axes $Ox_1$ and $Ox_2$) of an area of the surface of the elastic body $\Omega$ at the point $O$, when the displacement field $\mathbf{v}^{0*}(\mathbf{x})$ is defined as the solution of problem (3.1).

Thus, if in boundary condition (4.7) we make the replacement of coordinates

$$\xi_1 = \xi_1^\circ + \hat{\xi}_1, \quad \xi_2 = \xi_2^\circ + \hat{\xi}_2, \quad \xi_3 = \hat{\xi}_3$$

then the transformed boundary condition will depend only on the expression $\hat{\xi}_1^2 + \hat{\xi}_2^2$. Taking the axial symmetry of relation (4.4) into account, which in the main does not undergo any changes, the problem for determining the vector function $\mathbf{W}^*(\boldsymbol{\xi})$ turns out to be axisymmetrical, while the contact area $\omega^*$ turns out to be circular with centre at the point $(\xi_1^o, \xi_2^o)$ with coordinates determined by the first two formulae of (4.9).

Further, since when $|\hat{\xi}| \to \infty$ the expansion

$$\mathbf{T}(\boldsymbol{\xi}) = \mathbf{T}(\boldsymbol{\xi}) - \xi_1^\circ \frac{\partial \mathbf{T}}{\partial \xi_1}(\boldsymbol{\xi}) - \xi_2^\circ \frac{\partial \mathbf{T}}{\partial \xi_2}(\boldsymbol{\xi}) + O(\varepsilon|\boldsymbol{\xi}|^{-3})$$

holds, within the framework of the method or matched asymptotic expansions the outer asymptotic expansion (3.2) is made more precise by adding singular terms corresponding to the action on the boundary of the body $\Omega$ at the point $O$ of concentrated moments $\boldsymbol{M}_1 = \varepsilon^2 \boldsymbol{P} * \xi_2^o$ and $\boldsymbol{M}_2 = -\varepsilon^2 \boldsymbol{P} * \xi_1^o$. It is clear that these terms have no effect on the process of constructing the first correction. Note also that the appearance of concentrated moments at the point $O$ is the consequence of the displacement of the centre of the contact area $\omega_\varepsilon^*$ with respect to the point $O$.

Below, in order to keep the formulae simple, the "caps" over the centred coordinates will be omitted.

## 5. The axisymmetric problem of unilateral contact for the boundary layer

Let $a^*$ be the radius of contact $\omega^*$. On the basis of the above, boundary condition (4.7) can be rewritten in the form

$$\frac{\rho^2}{2R_0} - \Delta_0^* + W_\zeta^*(\rho, 0) + \frac{\sqrt{\varepsilon}}{2R_1}\rho^2 \frac{\partial W_\zeta^*}{\partial \zeta}(\rho, 0) + \frac{\sqrt{\varepsilon}}{R_2}\rho W_\rho^*(\rho, 0) = 0, \quad \rho \in [0, a^*] \tag{5.1}$$

Let us now return to the remaining relations (1.5) and (1.6) of the boundary condition of unilateral contact. It is not difficult to check that the unit vector of the outward normal $\mathbf{n}^\varepsilon(\rho)$ and the tangential vector in the radial direction $\mathbf{t}^\varepsilon(\rho)$ allow of expansions [in all cases below, $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_1(\rho)$]

$$\mathbf{n}^\varepsilon(\rho) = -\mathbf{e}_\zeta + \sqrt{\varepsilon}\boldsymbol{\Phi}_1' \mathbf{e}_\rho + O(\varepsilon), \quad \mathbf{t}^\varepsilon(\rho) = \mathbf{e}_\rho + \sqrt{\varepsilon}\boldsymbol{\Phi}_1' \mathbf{e}_\zeta + O(\varepsilon) \tag{5.2}$$

Using these expansions and proceeding in the same way as when deriving expansion (2.6), we obtain

$$\begin{aligned} &-\boldsymbol{\sigma}^{(n)}(\mathbf{W}^*; \rho, \sqrt{\varepsilon}\boldsymbol{\Phi}_1) = \\ &= \boldsymbol{\sigma}^{(\zeta)}(\mathbf{W}^*; \rho, 0) + \sqrt{\varepsilon}[\boldsymbol{\Phi}_1 \boldsymbol{\sigma}^{(\zeta)}(\mathbf{W}^*_{,\zeta}; \rho, 0) - \boldsymbol{\Phi}_1' \boldsymbol{\sigma}^{(\rho)}(\mathbf{W}^*; \rho, 0)] + \dots \end{aligned} \tag{5.3}$$

For the normal stress $\sigma_N^{(n)} = \boldsymbol{\sigma}^{(n)}\mathbf{n}^\varepsilon$ and the shear stress $\sigma_T^{(n)} = \boldsymbol{\sigma}^{(n)} t^\varepsilon$, expansions (5.2) and (5.3) will be

$$\sigma_N^{(n)}(\mathbf{W}^*; \rho, \sqrt{\varepsilon}\boldsymbol{\Phi}_1) = \sigma_{\zeta\zeta}(\mathbf{W}^*; \rho, 0) + \sqrt{\varepsilon}[\boldsymbol{\Phi}_1 \sigma_{\zeta\zeta}(\mathbf{W}^*_{,\zeta}; \rho, 0) - 2\boldsymbol{\Phi}_1' \sigma_{\rho\zeta}(\mathbf{W}^*; \rho, 0)] + \dots \tag{5.4}$$

$$\begin{aligned} &-\sigma_T^{(n)}(\mathbf{W}^*; \rho, \sqrt{\varepsilon}\boldsymbol{\Phi}_1) = \sigma_{\rho\zeta}(\mathbf{W}^*; \rho, 0) + \\ &+ \sqrt{\varepsilon}\{\boldsymbol{\Phi}_1 \sigma_{\rho\zeta}(\mathbf{W}^*_{,\zeta}; \rho, 0) - \boldsymbol{\Phi}_1'[\sigma_{\rho\rho}(\mathbf{W}^*; \rho, 0) - \sigma_{\zeta\zeta}(\mathbf{W}^*; \rho, 0)]\} + \dots \end{aligned} \tag{5.5}$$

Thus, the boundary condition expressing the absence of friction (between the touching surfaces of the elastic body and punch) and the shear loads on $\Gamma_c$ outside the contact area, according to expressions (1.5), (1.6) and (5.5), is taken on the unperturbed boundary as follows:

$$\sigma_{\rho\zeta}^0(\mathbf{W}^*) + \sqrt{\varepsilon}\{\boldsymbol{\Phi}_1 \sigma_{\rho\zeta}^0(\mathbf{W}^*_{,\zeta}) - \boldsymbol{\Phi}_1'[\sigma_{\rho\rho}^0(\mathbf{W}^*) - \sigma_{\zeta\zeta}^0(\mathbf{W}^*)]\} = 0, \quad \zeta = 0, \quad \rho \ge 0 \tag{5.6}$$

Note that the vector $\mathbf{V}^*(\varepsilon; \boldsymbol{\xi})$ on the boundary of the half-space $\zeta \ge 0$ satisfies the boundary conditions of no stresses and consequently has no influence when setting up relation (5.6).

Finally, the boundary condition $\sigma_N^{(n)}(\mathbf{u};\mathbf{x})$ when $\mathbf{x}\in\Gamma_c\backslash\omega_e$, expressing the absence of normal loads outside the contact area $\omega_\varepsilon$, according to relations (1.5) and (5.4), is shifted to the unperturbed boundary $\zeta=0$ as follows:

$$\sigma^0_{\zeta\zeta}(\mathbf{W}^*)+\sqrt{\varepsilon}[\Phi_1\sigma^0_{\zeta\zeta}(\mathbf{W}^*_{,\zeta})-2\Phi_1'\sigma^0_{\rho\zeta}(\mathbf{W}^*)] = 0,\quad \zeta = 0,\quad \rho\ge a^* \tag{5.7}$$

The construction of the inner asymptotic expansion $\mathbf{w}(\boldsymbol{\xi})$ according to representation (4.3) is reduced to searching for the vector function $\mathbf{W}^*(\boldsymbol{\xi})$ that satisfies, in the half-space $\xi_3\ge 0$, the equations of the Lamé system, on its boundary, the mixed boundary conditions (5.1), (5.6) and (5.7), and at infinity, the asymptotic condition (4.4).

For the contact pressure $p(\mathbf{u};\mathbf{x})=-\sigma_N^0(\mathbf{u};\mathbf{x})$ on the contact area $\omega_\varepsilon$, on changing to the extended coordinates (3.6), in view of the relation $\partial/\partial x_i=\varepsilon^{-1/2}\partial/\partial\xi_i$, we obtain the expression

$$p(\mathbf{u};\mathbf{x}) = -\varepsilon^{-1/2}\sigma_N^{(n)}(\mathbf{w};\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in\omega^*$$

Then, taking into account formulae (4.3) and (5.4), we find

$$p(\mathbf{u};\mathbf{x}) \simeq -\varepsilon^{-1/2}\{\sigma_{\zeta\zeta}(\mathbf{W}^*;\rho,0)+\sqrt{\varepsilon}[\Phi_1\sigma_{\zeta\zeta}(\mathbf{W}^*_{,\zeta};\rho,0)-2\Phi_1'\sigma_{\rho\zeta}(\mathbf{W}^*;\rho,0)]\} \tag{5.8}$$

As shown, the problem of unilateral contact for the boundary layer is axisymmetrical and structurally non-linear, since the radius $a^*$ of the contact area $\omega^*$ must be determined in the course of solving the problem from the condition that the contact pressure (5.8) vanishes at the edge of the contact area.

## 6. Solution of the problem for the boundary layer

The vector function $\mathbf{W}^*(\boldsymbol{\xi})$ will be constructed using the Papkovich–Neuber representation (2.11) together with the method of complex potentials.[29–32]

Making, in the Papkovich–Neuber representation (2.11), the substitution

$$2\mu\varphi = (1-2\nu)N+2(1-\nu)S,\quad 2\mu\psi = N_{,\zeta}+S_{,\zeta}$$

we obtain the following symmetrical representation[32]

$$\begin{aligned}2\mu W^*_\rho &= (1-2\nu)N_{,\rho}+2(1-\nu)S_{,\rho}+\zeta(N_{,\zeta\rho}+S_{,\zeta\rho})\\ 2\mu W^*_\zeta &= -2(1-\nu)N_{,\zeta}-(1-2\nu)S_{,\zeta}+\zeta(N_{,\zeta\zeta}+S_{,\zeta\zeta})\end{aligned} \tag{6.1}$$

Here, the components of the stresses are expressed as follows:

$$\begin{aligned}\sigma_{\rho\rho}(\mathbf{W}^*) &= 2\nu\rho^{-1}(N+S)_{,\rho}+(N+2S)_{,\rho\rho}+\zeta(N+S)_{,\zeta\rho\rho}\\ \sigma_{\zeta\zeta}(\mathbf{W}^*) &= -N_{,\zeta\zeta}+\zeta(N+S)_{,\zeta\zeta\zeta},\quad \sigma_{\rho\zeta}(\mathbf{W}^*) = S_{,\rho\zeta}+\zeta(N+S)_{,\zeta\zeta\rho}\end{aligned} \tag{6.2}$$

We emphasize that expressions (6.2) are transformed taking into account Laplace's equation, which should be satisfied by the harmonic potentials $N$ and $S$.

We will assume that

$$N = \mathrm{Re}\int_0^{a^*} n(t)\ln\kappa(t)dt,\quad \kappa(t) = \zeta+it+\sqrt{(\zeta+it)^2+\rho^2} \tag{6.3}$$

$$S = \sqrt{\varepsilon}(c_1S^1+S^2) \tag{6.4}$$

$$S^1 = \mathrm{Re}\int_0^{a^*} n(t)\left((\zeta+it)\ln\frac{\kappa(t)}{2R_1}+\zeta+it-\kappa(t)\right)dt,\quad S^2 = \mathrm{Im}\int_0^{a^*} s^2(t)\ln\kappa(t)dt \tag{6.5}$$

The densities $n(t)$ and $s^2(t)$ and the coefficient $c_1$ are determined by substituting expressions (6.1)–(6.5) into the boundary conditions (5.1), (5.6) and (5.7).

We will introduce the notation

$$\tilde{n} = \int_0^{a*} n(t)dt, \quad \mathcal{N}(\rho) = \int_\rho^{a*} \frac{n(t)tdt}{\sqrt{t^2-\rho^2}}$$

Indirect calculations with $\zeta = 0$ show that

$$\sigma^0_{\zeta\zeta}(N) = \sigma^0_{\rho\zeta}(N) = 0, \quad \sigma^0_{\zeta\zeta}(N_{,\zeta}) = \sigma^0_{\rho\zeta}(N_{,\zeta}) = 0, \quad \rho \ge a*$$
$$\sigma^0_{\rho\rho}(N) = -\frac{1-2\nu}{\rho}N^0_{,\rho} = -\frac{1-2\nu}{\rho^2}\tilde{n}, \quad \rho \ge a* \tag{6.6}$$

$$\sigma^0_{\rho\zeta}(N_{,\zeta}) = N^0_{,\zeta\zeta\rho} = (\rho^{-1}\mathcal{N}'(\rho))', \quad \rho < a* \tag{6.7}$$

$$\sigma^0_{\rho\rho}(N) - \sigma^0_{\zeta\zeta}(N) = -\frac{1-2\nu}{\rho}N^0_{,\rho} = -\frac{1-2\nu}{\rho^2}(\tilde{n} - \mathcal{N}(\rho)), \quad \rho < a* \tag{6.8}$$

Boundary condition (5.7) gives

$$\sigma^0_{\zeta\zeta}(S) = 0, \quad \rho \ge a* \tag{6.9}$$

Condition (6.9) is satisfied for the potential (6.4) by construction.

Boundary condition (5.6) must be considered separately on the contact area (when $\rho < a*$) and outide it (when $\rho \ge a*$). Thus, taking the equality $\sigma^0_{\rho\zeta}(N) = 0$ and expressions (6.7) and (6.8) into account, we will have

$$\sigma^0_{\rho\zeta}(c_1S^1 + S^2) = \Phi'_1\sigma^0_{\rho\rho}(N), \quad \rho \ge a* \tag{6.10}$$

$$\sigma^0_{\rho\zeta}(c_1S^1 + S^2) = -\Phi_1\sigma^0_{\rho\zeta}(N_{,\zeta}) + \Phi'_1[\sigma^0_{\rho\rho}(N) - \sigma^0_{\zeta\zeta}(N)], \quad \rho < a* \tag{6.11}$$

Then, for the potential $S^1$ we obtain

$$\sigma^0_{\rho\zeta}(S^1) = \rho^{-1}\tilde{n}, \quad \rho \ge a*; \quad \sigma^0_{\rho\zeta}(S^1) = \rho^{-1}\tilde{n} - \rho^{-1}\mathcal{N}(\rho), \quad \rho < a* \tag{6.12}$$

Now, substituting the final expression into boundary condition (6.10) and comparing it with the first expression of (6.12), taking relations (4.1) into account, we find

$$c_1 = -(1-2\nu)/R_1 \tag{6.13}$$

Thus, substituting representation (6.4) into boundary condition (6.11) and taking formulae (6.7), (6.8) and (6.12) into account, we derive the following boundary condition

$$\sigma^0_{\rho\zeta}(S^2) = -\rho^2(2R_1)^{-1}(\rho^{-1}\mathcal{N}'(\rho))', \quad \rho < a* \tag{6.14}$$

We will denote the right-hand side of equality (6.14) by $q^2(\rho)$. Then, on the basis of the solution of Abel's integral equation (for details, see, for example, Ref. 32, Section 9.2), we can represent the density of integral $S^2$ in the form

$$s^2(t) = \frac{2t}{\pi}\int_t^{a*} \frac{q^2(x)}{\sqrt{x^2-t^2}}dx \tag{6.15}$$

Below we will need the value of the vertical displacement [see the second formula of (6.1)]

$$\frac{2\mu}{1-2\nu}U^0_\zeta(S^2) = -S^{20}_{,\zeta} = \int_\rho^{a*} \frac{s^2(t)dt}{\sqrt{t^2-\rho^2}}, \quad \rho < a* \tag{6.16}$$

We substitute expression (6.15) into the right-hand side of relation (6.16) and change the order of integration. As a result of simple calculations, we obtain

$$U_\zeta^0(S^2) = \frac{1-2\nu}{2\mu}\int_\rho^{a*} q^2(x)dx, \quad \rho < a*$$

Now, substituting here the expression $q^2(x)$ from the right-hand side of equality (6.14), and integrating by parts, we obtain

$$\frac{4\mu R_1}{1-2\nu}U_\zeta^0(S^2) = -a*\mathcal{N}'(a*) + \rho\mathcal{N}'(\rho) - 2\mathcal{N}(\rho) \tag{6.17}$$

Then, similar to the left-hand equation of system (6.16), we obtain

$$-\frac{2\mu}{1-2\nu}U_\zeta^0(S^1) = S_{,\zeta}^{10} = \mathrm{Re}\int_0^{a*} n(t)\ln\frac{it+\sqrt{\rho^2-t^2}}{2R_1}dt =$$
$$= \ln\frac{\rho}{2R_1}\int_0^\rho n(t)dt + \int_\rho^{a*} n(t)\ln\frac{t+\sqrt{t^2-\rho^2}}{2R_1}dt \tag{6.18}$$

Thus, the solution of the problem of unilateral contact for the boundary layer is expressed in terms of the complex potentials (6.3) and (6.5), which are determined by the density $n(t)$. To find it, we substitute expressions (6.1) and (6.4) into the remaining boundary condition (5.1). With the same accuracy as that with which relation (5.1) was obtained, we will have

$$\frac{\rho^2}{2R_0} - \Delta_0^* + U_\zeta^0(N) + \sqrt{\varepsilon}\left\{c_1U_\zeta^0(S^1) + U_\zeta^0(S^2) + \frac{\rho^2}{2R_1}\frac{\partial U_\zeta}{\partial\zeta}(N)\bigg|_{\zeta=0} + \frac{\rho}{R_2}U_\rho^0(N)\right\} = 0$$
$$\rho \le a* \tag{6.19}$$

and here (in the next three formulae it is assumed that $\rho < a*$)

$$-(2\pi)^{-1}\vartheta^{-1}U_\zeta^0(N) = \int_0^\rho \frac{n(t)tdt}{\sqrt{\rho^2-t^2}}$$
$$(\pi\beta\vartheta)^{-1}U_\rho^0(N) = \rho^{-1}\tilde{n} - \rho^{-1}\mathcal{N}(\rho), \quad -(\pi\beta\vartheta)^{-1}U_{\zeta,\zeta}(N)|_{\zeta=0} = \rho^{-1}\mathcal{N}'(\rho) \tag{6.20}$$

When $\sqrt{\rho^2+\zeta^2} \to \infty$, the following asymptotic relation holds

$$S^1 \sim \tilde{n}(\zeta\ln[(\zeta+\sqrt{\zeta^2+\rho^2})/(2R_1)] - \sqrt{\zeta^2+\rho^2})$$

The asymptotic matching condition (4.4) will be satisfied if the following normaligation condition is satisfied

$$2\pi\tilde{n} = -P* \tag{6.21}$$

Eq. (6.21) enables us to establish the relation between the force $P*$ and the displacement $\delta_0^*$. Here, the radius $a*$ of the contact area $\omega*$ is determined by the condition for the contact pressure (5.8) at its edge to vanish.

Denoting the expression in braces in formula (5.8) by $-p*(\rho)$,we will have

$$-p*(\rho) = \sigma_{\zeta\zeta}^0(N) + \sqrt{\varepsilon}\Phi_1\sigma_{\zeta\zeta}^0(N_{,\zeta}) \tag{6.22}$$

Differentiating both sides of the second formula of (6.2), we obtain $\sigma_{\zeta\zeta}(N_{,\zeta}) = \zeta N_{,\zeta\zeta\zeta\zeta}$. Since direct differentiation with respect to $\zeta$ of integral (6.3) leads to integrals which diverge in the limit as $\zeta \to 0$, we will use the formula[32]

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{1}{\sqrt{(\zeta+it)^2+\rho^2}} \equiv \frac{1}{\zeta+it}\frac{\partial}{\partial\zeta}\frac{1}{\sqrt{(\zeta+it)^2+\rho^2}}$$

Using this formula three times, we obtain

$$N^0_{,\zeta\zeta\zeta\zeta} = 3\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^2 \mathcal{N}(\rho) - \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^3 \int_\rho^{a*} \frac{n(t)t^3 dt}{\sqrt{t^2-\rho^2}}$$

Now, taking into account that as $\varepsilon \to 0$ the density $n(t)$ converges to the Hertzian, we conclude that the integrals in the expression for $N^0_{,\zeta\zeta\zeta\zeta}$ in the principal term of the asymptotic form will be polynomials in $\rho^2$. Consequently, within the framework of the accuracy of the asymptotic constructions, the second term in sum (6.22) vanishes, and we finally have

$$p*(\rho) = \rho^{-1}\mathcal{N}'(\rho) \tag{6.23}$$

Note that the previous discussion also establishes the equality $\mathcal{N}'(a*) = 0$.

We will also show that the condition $n(a^*) = 0$, follows from the condition $p^*(a^*) = 0$. In fact, from formula (6.23) we find

$$0 = \frac{1}{a*}\lim_{\Delta\rho \to 0-}\frac{1}{\Delta\rho}\int_{a*+\Delta\rho}^{a*}\frac{n(t)tdt}{\sqrt{t^2-(a*+\Delta\rho)^2}}$$

Now, applying the theorem on the average value of the integral (the continuity of the function $n(t)$ at the point $a^*$ is proved indirectly), we arrive at the limit

$$0 = \lim_{\Delta\rho \to 0-}\frac{1}{\Delta\rho}n(a*+\theta\Delta\rho)\sqrt{2a*(-\Delta\rho)}$$

where $(0, 1) \ni \theta$ is a certain number (depending on $\Delta\rho$). Hence, it immediately follows that the solution of integral equation (6.19) should vanish at the edge of the contact area, which, we recall, yields an equation for determining the radius $a^*$.

## 7. The asymptotic form of the resultant contact pressure

Transforming the first formula of system (6.20), for the density of the integral (6.3) we derive the representation

$$n(t) = -\pi^{-2}\vartheta^{-1}\frac{d}{dt}\int_0^t \frac{\rho U^0_\zeta(N;\rho)d\rho}{\sqrt{t^2-\rho^2}} \tag{7.1}$$

We will put

$$n(t) = n_0(t) + \sqrt{\varepsilon}n_1(t) \tag{7.2}$$

$$a* = a_0^* + \sqrt{\varepsilon}a_1^* \tag{7.3}$$

The density $n_0(t)$ is obviously determined in accordance with Hertz theory

$$n_0(t) = -\pi^{-2}\vartheta^{-1}(\delta_0^* - v_3^{0*}(O) - R_0^{-1}t^2) \tag{7.4}$$

From the condition $n(a^*) = 0$ in the limit as $\varepsilon \to 0$ it follows that $n_0(a_0^*) = 0$; consequently

$$a_0^* = ((\delta_0^* - v_3^{0*}(O))R_0)^{1/2} \tag{7.5}$$

From Eq. (6.21) we obtain the expansion

$$-\frac{1}{2\pi}P* = \int_0^{a_0^*} n_0(t)dt + \sqrt{\varepsilon}\left(\int_0^{a_0^*} n_1(t)dt + n_0(a_0^*)a_1^*\right)$$

Since $n_0(a_0^*) = 0$, the asymptotic form of the equivalent contact pressure is sought irrespective of the solution of the problem of the variation of the contact area, which consists of determining the correction in expansion (7.3), i.e.

$$P^* = -2\pi\int_0^{a_0^*}(n_0(t)+\sqrt{\varepsilon}n_1(t))dt \tag{7.6}$$

As before, from formula (6.23) we obtain

$$p^*(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\int_\rho^{a_0^*}\frac{n_0(t)+\sqrt{\varepsilon}n_1(t)}{\sqrt{t^2-\rho^2}}dt \tag{7.7}$$

Formula (7.7) determines the outer asymptotic expansion for the contact pressure density that holds at a distance from the boundary of the contact area.

We find the density $n_1(t)$ by substituting into the integral (7.1), instead of $U_\zeta^0(N;\rho)$, the sum of the expression $P_0^*(\tilde{C}_\varepsilon + \tilde{A}_0)$ and the expression in braces in formula (6.19), taken with the opposite sign. For this purpose, we calculate the terms in the density $n_0(t)$ occurring in it.

Thus, from formula (6.18) we obtain

$$3\pi\beta^{-1}R_0U_\zeta^0(S^1) = 2a_0^{*3}\ln\frac{a_0^*+h_0^*(\rho)}{2R_1}+\frac{1}{3}h_0^{*3}(\rho)+(\rho^2-3a_0^{*2})h_0^*(\rho) \tag{7.8}$$

where

$$h_0^*(\rho) = \sqrt{a_0^{*2}-\rho^2}$$

Using formula (6.17), taking into account the quantities $\mathcal{N}(a_0^*) = 0$ and Eq. (7.4), we will have

$$3\pi\beta^{-1}R_1R_0U_\zeta^0(S^2) = (2a_0^{*2}+\rho^2)h_0^*(\rho) \tag{7.9}$$

The third and second formulae of (6.20) respectively yield

$$\frac{\rho^2}{2R_1}U_{\zeta,\zeta}(N)\Big|_{\zeta=0} = -\frac{\beta}{\pi}\frac{\rho^2}{R_1R_0}h_0^*(\rho), \quad -\frac{3\pi}{2\beta}R_0\rho U_\rho^0(N) = a_0^{*3}-h_0^{*3}(\rho) \tag{7.10}$$

It is then easy to show that the following equality holds

$$\int_0^{a_0^*}n_1(t)dt = -\pi^{-2}\vartheta^{-1}a_0^*\int_0^1\frac{u_1(a_0^*t)t}{\sqrt{1-t^2}}dt \tag{7.11}$$

where, by relations (6.13), (2.15), (3.10), and (7.8)–(7.10), we have

$$\begin{aligned}\frac{3\pi}{2\beta}R_1R_0u_1(\rho) &= (1-2\nu)\left(a_0^{*3}\ln\frac{a_0^*+h_0^*(\rho)}{2R_1}-\frac{1}{3}h_0^{*3}(\rho)-a_0^{*2}h_0^*(\rho)\right)+\\ &+\frac{R_1}{R_2}(a_0^{*3}-h_0^{*3}(\rho))-h_0^{*3}(\rho)-\frac{2}{\beta}R_1a_0^{*3}(C_\varepsilon+A_0)\end{aligned} \tag{7.12}$$

Substituting expression (7.12) into formula (7.11), after elementary integration, we find [see formula (7.6)]

$$P^* = P_0^*+\sqrt{\varepsilon}P_1^*, \quad P_0^* = (12\pi)^{-1}\vartheta^{-1}R_0^{-1}a_0^{*3} \tag{7.13}$$

$$P_1^* = P_0^*\frac{\beta a_0^*}{\pi R_1}\left\{(1-2\nu)\left(\ln\frac{2\sqrt{\varepsilon}a_0^*}{R_1}-\frac{19}{12}\right)+\frac{3R_1}{4R_2}-\frac{1}{4}-\frac{2}{\beta}R_1A_0\right\} \tag{7.14}$$

Formulae (7.13), (7.14), (3.10) and (7.5) give the asymptotic form of the resultant contact pressure as a function of the displacement of the punch $\delta_0$. The inverse relation is also of interest. With the accuracy with which the relations (7.13) and (7.14) were obtained, we will have

$$\delta_0 - v_3^0(O) = \left(\frac{3(1-\nu)}{8\mu}\frac{P}{\sqrt{R_0}}\right)^{2/3} + \\ + \frac{1-2\nu}{4\pi\mu}\frac{P}{R_1}\left\{(1-2\nu)\left(\frac{1}{3}\ln\frac{\mu R_1^3}{3(1-\nu)R_0P} + \frac{19}{12}\right) - \frac{3R_1}{4R_2} + \frac{1}{4} + \frac{2(1-\nu)}{1-2\nu}R_1A_0\right\} \tag{7.15}$$

The form of the functional relation (7.15) agrees with the corresponding result obtained in Ref. 33 (see formula (5.18)).

Note that it is not difficult to write in explicit form the outer asymptotic expansion for the contact pressure density (7.7), where the function $n_1(t)$ is defined in terms of the function (7.12) by means of the formula

$$n_1(t) = -\frac{2\mu}{\pi(1-\nu)}\frac{d}{dt}\int_0^t \frac{\rho u_1(\rho)}{\sqrt{t^2-\rho^2}}d\rho \tag{7.16}$$

The approximate expression for the contact pressure density in closed form, suitable over the entire contact area, can be obtained by the method described in Ref. 9.

## 8. The variation of the radius of the contact area

From the condition $n(a^*) = 0$, taking expansions (7.2) and (7.3) and the equality $n(a_0^*) = 0$ into account, we obtain

$$a_1^* = -\frac{n_1(a_0^*)}{n_0'(a_0^*)} \tag{8.1}$$

where, in view of expression (7.4),

$$n_0'(a_0^*) = \frac{2\mu}{\pi(1-\nu)}\frac{2a_0^*}{R_0} \tag{8.2}$$

We will calculate the function $n(\rho)$ when $\rho = a_0^*$. In the integral (7.16) we will replace the variable of integration using the formula $x = \sqrt{1-\tau^2}$ and differentiate the integrand with respect to the parameter (this operation is legitimate as the functions in the integrand obtained are continuous in the segment [0, 1]). As a result, we will have

$$n_1(a_0^*) = -\frac{2\mu}{\pi(1-\nu)}\int_0^1 \frac{\tau u_1(a_0^*\tau) + \tau^2 u_1'(a_0^*\tau)}{\sqrt{1-\tau^2}}d\tau \tag{8.3}$$

Further we differentiate expression (7.12), simplify the result and then substitute $t = a_0^*\tau$. We will have

$$\frac{3}{2\beta}\frac{R_1R_0}{{a_0^*}^2}u_1'(a_0^*\tau) = (1-2\nu)\tau((1+x)^{-1}+x) + 3\left(\frac{R_1}{R_2}+1\right)\tau x; \quad x = \sqrt{1-\tau^2} \tag{8.4}$$

Now, substituting expressions (7.12) and (8.4) into formula (8.3) and carrying out elementary integration, we find the quantity $n_1(a_0^*)$, the substitution of which into formula (8.1), taking equality (8.2)into account, finally yields

$$a_1^* = \frac{1-2\nu}{3\pi(1-\nu)}\frac{{a_0^*}^2}{R_1}\left\{-\frac{2(1-\nu)}{1-2\nu}R_1(C_\varepsilon + A_0) + \\ + (1-2\nu)\left(\ln\frac{a_0^*}{2R_1} + 2\ln 2 - \frac{5}{6}\right) + \frac{3R_1}{2R_2} + \frac{1}{2}\right\} \tag{8.5}$$

To check this result, we will take the limit on the right-hand part of relation (8.5) as $R_1 \to \infty$. In the case of a plane boundary of an elastic body, formula (8.5) becomes

$$a_1^* = \frac{1-2\nu}{2\pi(1-\nu)}\frac{a_0^{*2}}{R_2}\left\{1-\frac{4}{3}\frac{(1-\nu)}{(1-2\nu)}A_0R_2\right\} \tag{8.6}$$

Formula (7.3), taking expressions (8.6) and (7.5) into account, where $R_0 = R_2$, agrees with formula (4.11) of Ref. 11 and formula (6.1) of Ref. 9, which were obtained in a different way.

Note that, in the limiting case examined, the result of Ref. 11 for the radius of the contact area is not obtained exactly, since the formulation of the initial contact problem takes into account the tangential displacements on the surface of the elastic body in the contact area (see relations (1.4)–(1.6)).

## 9. Conclusions

We recall that, according to the Hertz theory,[34] when calculating local contact pressures, an elastic body can be regarded as an elastic half-space (see, in particular, Ref. 2, Section 4.2). Here, neither the shape of the elastic body outside the region of local perturbations nor the conditions of its fixing are considered in the calculation.

In the present paper, an asymptotic model is constructed that extends Hertz theory in the axisymmetric case, and, more precisely, when the surface of the punch and the surface of the elastic body in the vicinity of the contact area are approximated by paraboloids of revolution. The proposed asymptotic model includes the coefficient of local compliance $A_0$, which comprises the integral characteristics of the geometry of the elastic body and its fixing conditions.

We emphasize that the extension of the Hertz theory that takes into account the boundedness of the elastic body required a more precise formulation of the contact conditions, as the corresponding corrections turn out to be of the first order.

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## References

1. Kravchuk AS. The Hertz problem for linearly and non-linearly elastic bodies of finite dimensions. *Dokl Akad Nauk SSSR* 1976;**230**(2):308–10.
2. Johnson KL. *Contact Mechanics*. Cambridge: University Press; 1985.
3. Shtayerman IYa. The solution of contact problems with a known Green's function. *Prikl Mat Mekh* 1941;**5**(3):409–18.
4. Shtayerman IYa. *The Contact Problem of the Theory of Elasticity*. Moscow and Leningrad: Gostekhizdat; 1949.
5. Mow VC, Chow PL, Long FF. Microslips between contacting paraboloids. *Trans ASME J Appl Mech* 1967;**34**(2):321–8.
6. Korolev AA. Elastic contact of smooth bodies of complex shape. *Izv Ross Akad Nauk MTT* 2002;**3**:59–71.
7. Kravchuk AS. Solution of contact problems with a known Green's function. *Prikl Mat Mekh* 1982;**46**(2):283–8.
8. Rabinovich VL, Sipcic SR, Sarin VK. Three-dimensional unilateral frictionless contact problem for finite bodies. *Trans ASME J Appl Mech* 1994;**61**(1):54–9.
9. Argatov II. Approximate solution of the axisymmetric contact problem allowing for tangential displacements on the contact surface. *Zh Prikl Mekh Tekh Fiz* 2004;**45**(1):143–50.
10. Galanov BA., Krivonos, Yu.M. Allowance in the Hertz problem for tangential displacements on the contact surface. In *Computer and Applied Mathematics*. Kiev: Vishcha Shkola; 1984, No. **53**:87–94.
11. Argatov II. The indentation of a punch in the form of an elliptic paraboloid into the plane boundary of an elastic body. *Prikl Mat Mekh* 1999;**63**(4):671–9.
12. Bondareva VF. Contact problems for an elastic sphere. *Prikl Mat Mekh* 1971;**35**(1):61–70.
13. Karpenko VA. Axisymmetric indentation of two punches into an elastic sphere. *Prikl Mat Mekh* 1976;**40**(4):763–6.
14. Aleksandrov VM. The approximate solution of one type of integral equations. *Prikl Mat Mekh* 1962;**26**(5):934–43.
15. Aleksandrov VM, Pozharskii DA. *Unclassical Three-dimensional Problems of the Mechanics of Contact Interactions of Elastic Bodies*. Moscow: Faktorial; 1998.
16. Van Dyke MD. *Perturbation Methods in Fluid Mechanics*. New York and London: Academic Press; 1964.
17. Il'in AM. *Matching of Asymptotic Expansions of Solutions of Boundary-value Problems*. Moscow: Nauka; 1989.

18. Argatov II, Nazarov SA. The method of matched expansions for problems with small contact areas. In: *Mechanics of Contact Interactions*. Moscow: Fizmatlit; 2001. p. 73–82.
19. Argatov II. Improvement of the asymptotic solution obtained by the method of matched expansions in the contact problem of the theory of elasticity. *Zh Vychisl Mat Mat Fiz* 2000;**40**(4):623–32.
20. Nowacki W. *Theory of Elasticity*. Warsaw: PWN; 1970.
21. Guz' AN, Nemish YuN. *Method of Boundary Shape Perturbation in Continuum Mechanics*. Kiev: Vishcha Shkola; 1989.
22. Banichuk NV. Determination of the shape of a curved crack by the small-parameter method. *Izv Akad Nauk SSSR MTT* 1970;**2**:130–7.
23. Movchan AV, Nazarov SA, Polyakova OR. The quasistatic growth of a semi-infinite crack in a plane containing small defects. *C R Acad Sci Paris Sér 2* 1991;**313**(11):1223–8.
24. Sternberg E, Rosenthal F. The elastic sphere under concentrated loads. *J Appl Mech* 1952;**19**(4):413–21.
25. Argatov II. The characteristics of the local compliance of an elastic body under the action on a plane area of its boundary of a small punch. *Zh Prikl Mekh Tekh Fiz* 2002;**43**(1):177–85.
26. Love AEH. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge: University Press; 1927.
27. Arutyunyan NKh, Movchan AB, Nazarov SA. A procedure for solving problems of the theory of elasticity in unbounded regions with paraboloidal and cylindrical inclusions or cavities. *Uspekhi Mekhaniki* 1987;**10**(4):3–91.
28. Argatov II. Asymptotic solution of the contact problem for a three-dimensional elastic body of finite dimensions. *Prikl Mat Mekh* 1999;**63**(6):1001–7.
29. Rostovtsev NA. Complex stress functions in the axisymmetric contact problem of the theory of elasticity. *Prikl Mat Mekh* 1953;**17**(5):611–4.
30. Rostovtsev NA. Complex potentials in the problem of a punch that is circular in plan. *Prikl Mat Mekh* 1957;**21**(1):77–82.
31. Green AE, Zerna W. *Theoretical Elasticity*. Oxford: Clarendon Press; 1954.
32. Hills DA, Nowell D, Sackfield A. *Mechanics of Elastic Contact*. Oxford: Butterworth-Heinemann; 1993.
33. Argatov II. Approximate solution of the axisymmetric contact problem for an elastic sphere. *Prikl Mat Mekh* 2005;**69**(2):303–14.
34. Hertz H. Über die Berührung fester elastischer Körper. *J Reine Angew Math* 1882;**92**:156–71.


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